More about 3-d graphs.

$$x^2 + y^2 + 4z^2 = 64$$

Techniques: (1) Make one of the variables 0, then graph as a 2-d graph.

(2) Make that same variable something else (eg 1, 2, 1.5). Then make the 2-d graph.

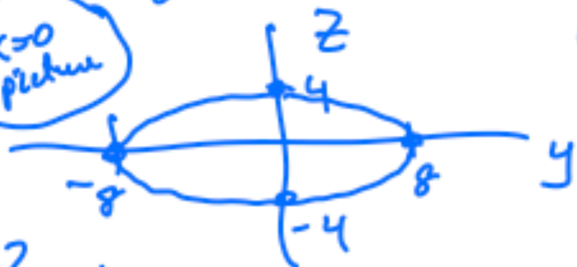
Graphing by slices.

$$x^2 + y^2 + 4z^2 = 64$$

Let $x = 0$ $\quad y^2 + 4z^2 = 64$

$y^2 = 64$
$y = \pm 8$

x=0 picture



$4z^2 = 64$
$z^2 = 16$
$z = \pm 4$

Let $x = 2$ $\quad 4 + y^2 + 4z^2 = 64$

$$y^2 + 4z^2 = 60$$

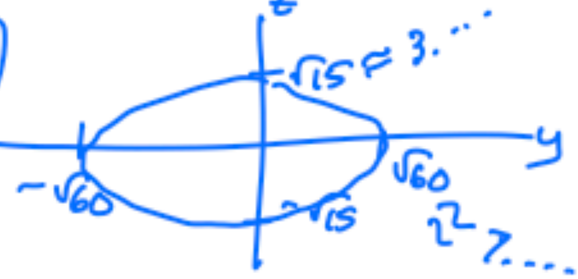
if $z=0$ $\quad y^2=60 \Rightarrow y=\pm\sqrt{60}$

if $y=0$ $\quad 4z^2=60 \Rightarrow z^2=15$

$\Rightarrow z=\pm\sqrt{15}$

also $x=-2$

$x=2$ picture



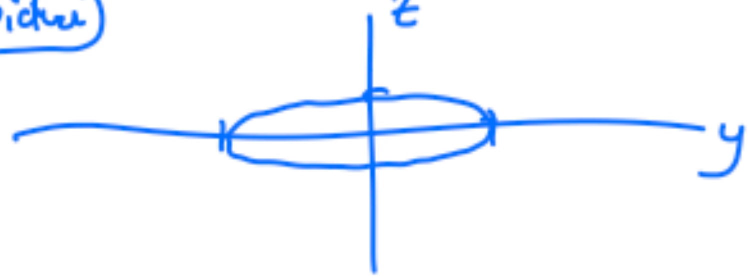
$x=6$ picture:
$$36+y^2+4z^2=64$$
$$y^2+4z^2=28$$

$z=0$ $\quad y=\pm\sqrt{28} \approx 5.\ldots$

$y=0$ $\quad 4z^2=28 \Rightarrow z^2=7 \Rightarrow z=\pm\sqrt{7} \approx 2.\ldots$

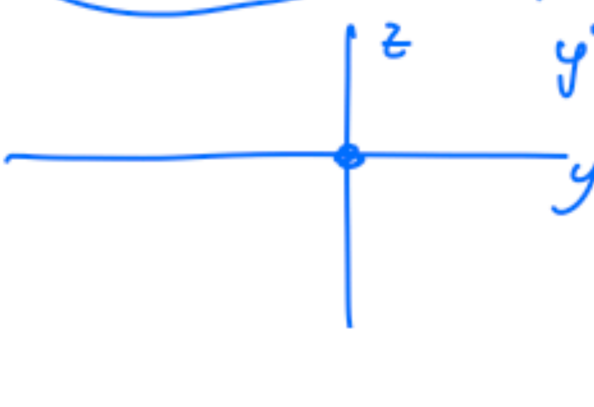
also $x=-6$

$x=6$ picture



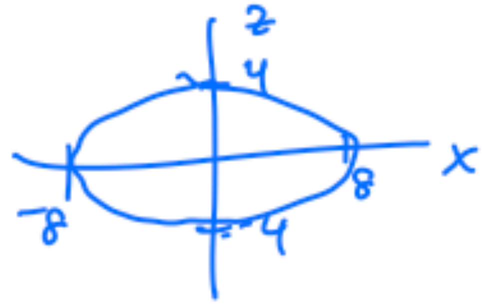
$x=\pm 8$ picture $\quad 64+y^2+4z^2=64$

$$y^2+4z^2=0$$
$$\Rightarrow (y,z)=(0,0)$$



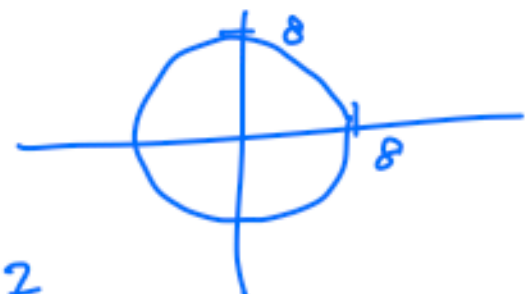
Now let's check $y=0$

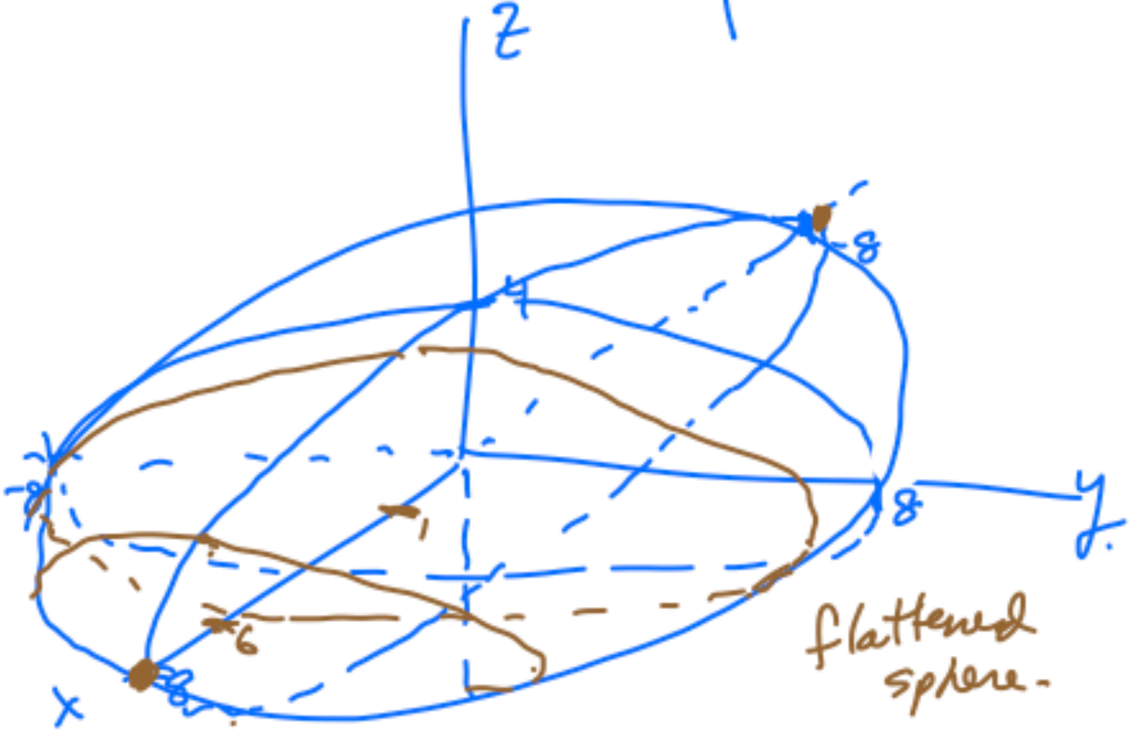
$$x^2+0+4z^2=64$$



$z=0$

$$x^2+y^2+0=64$$





flattened sphere.

If we graph $x^2+y^2+z^2=64$,
we would get a sphere of radius 8.

[Notice $x^2+y^2+z^2=\left(\begin{array}{c}\text{dist from}\\ (x,y,z) \text{ to } (0,0,0)\end{array}\right)^2=64$

$\iff$ distance from (x,y,z) to $(0,0,0)$ is 8

So the graph is the set of (x,y,z) that are 8 units from the origin.

$\Rightarrow$ Sphere of radius 8.

Let's generalize:

$$(x-a)^2+(y-b)^2+(z-c)^2=R^2$$

where $a,b,c,R>0$ are constants,

$\iff$ (distance from (x,y,z) to (a,b,c))$^2=R^2$

$\Longleftrightarrow$ (x,y,z) is on the sphere centered at (a,b,c) of radius R.

Greater generalization:

we had $$\frac{(x-a)^2}{R^2} + \frac{(y-b)^2}{R^2} + \frac{(z-c)^2}{R^2} = 1$$

Ellipsoid

$$\frac{(x-a)^2}{R_1^2} + \frac{(y-b)^2}{R_2^2} + \frac{(z-c)^2}{R_3^2} = 1$$

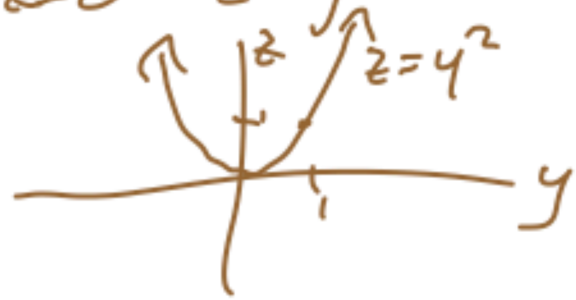
R_1 = radius in x-direction
R_2 = radius in y-direction
R_3 = " z-direction.

Example $z = x^2 + y^2$

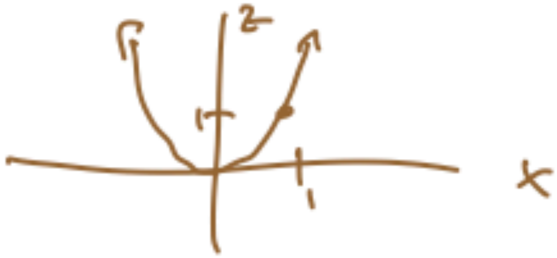
Slices: Let $z=0 \Rightarrow x^2+y^2=0$

$(0,0,0)$. ✓
the only part of the graph in the xy plane.

Let $x=0$ $\quad z=y^2$

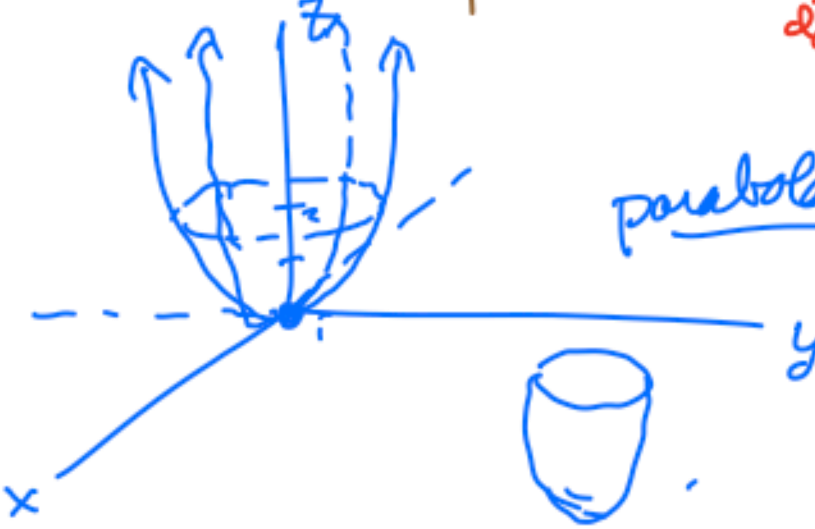


Let $y=0$ $\quad z=x^2$



Let $z=2$ $\quad 2=x^2+y^2$

Note if $z<0$, no graph $x^2+y^2=z$ does not work.



paraboloid.

Slight variation.

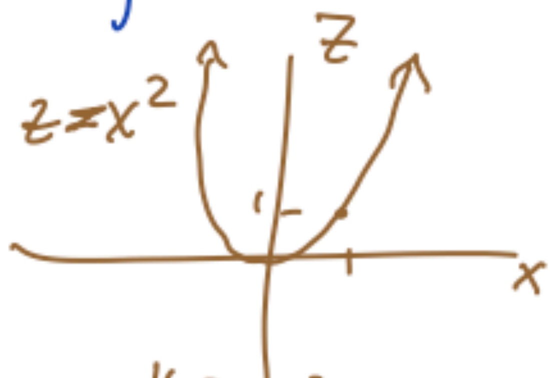
$$z = x^2 + 4y^2$$

Elliptic Paraboloid

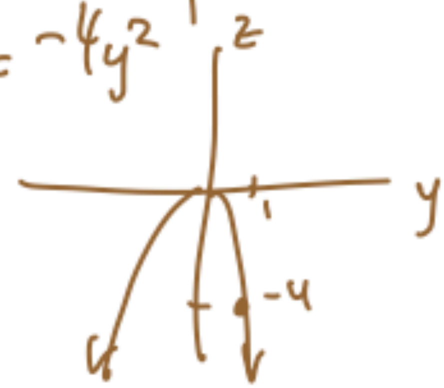
↑ squished by factor of 2 in y-direction

$$z = x^2 - 4y^2$$

Slices: $y = 0 \Rightarrow z = x^2$



$x = 0 \Rightarrow z = -4y^2$

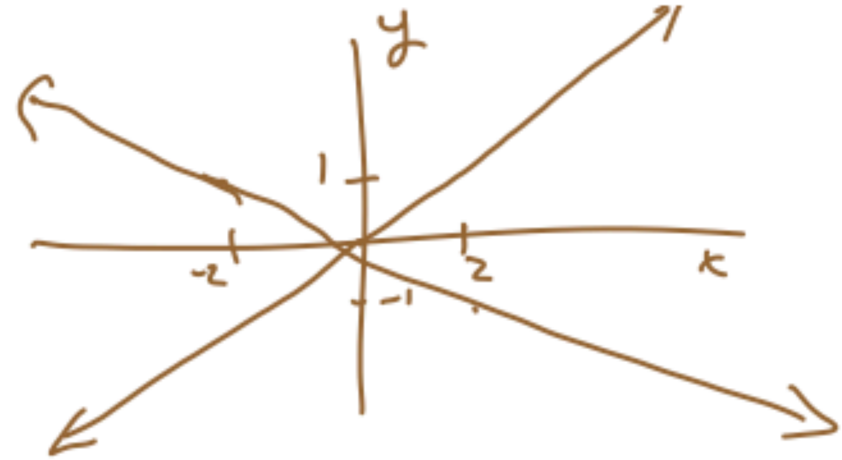


$z = 0 \iff x^2 - 4y^2 = 0 \iff (x+2y)(x-2y) = 0$

$x + 2y = 0$ or $x - 2y = 0$

$y = -\frac{1}{2}x \qquad y = \frac{1}{2}x.$

$z = 0$



$z = 4 \qquad 4 = x^2 - 4y^2$

$$1 = \frac{x^2}{4} - y^2$$

$z = -4 \qquad -4 = x^2 - 4y^2$

$$1 = -\frac{x^2}{4} + y^2$$

What is $z = x^2 - 4y^2$?

Saddle surface

$x^2 + y^2 + z^2 = 4$ sphere of radius 2.

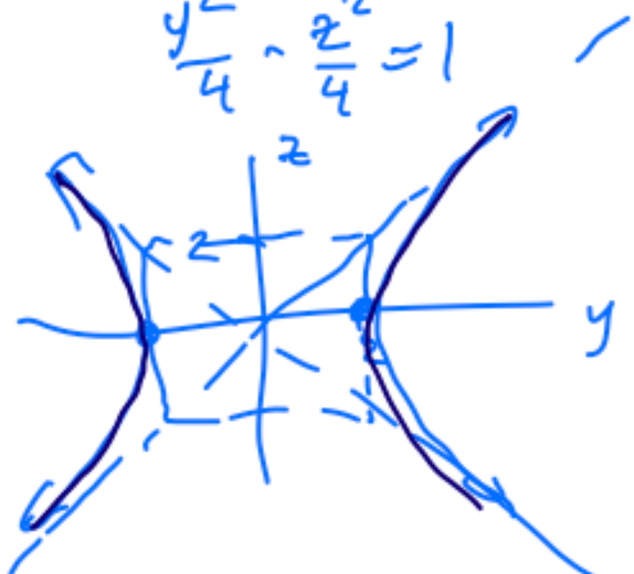
What is

$x^2 + y^2 - z^2 = 4$?

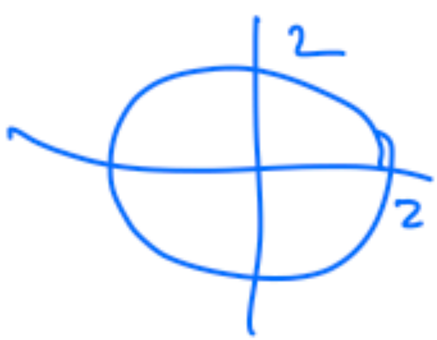
Slices:

$x = 0$ $y^2 - z^2 = 4$

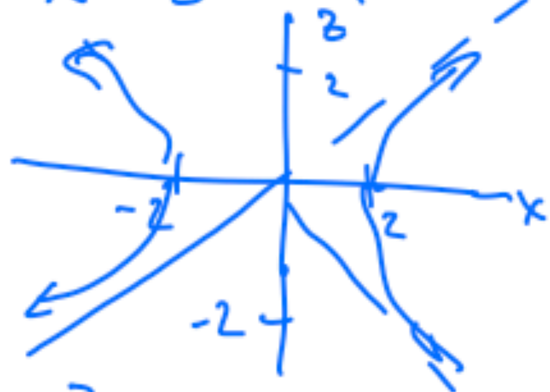
$\frac{y^2}{4} - \frac{z^2}{4} = 1$



$z = 0$ $x^2 + y^2 = 4$



$y = 0 \qquad x^2 - z^2 = 4$



hyperboloid of one sheet.

other examples

2 - sheeted hyperboloid.

$z^2 - x^2 - y^2 = 4$

$z^2 + x^2 - 4y^2 = 4$
elliptic hyperboloid of one sheet.

same as $z^2 + x^2 - y^2 = 4$
squished by factor of 2
in y-direction.

Doing Calculus in Higher Dimensions.

First example: curves in $\mathbb{R}^n$.

In $\mathbb{R}^2$, we often looked at curves like $y=x^2$ or $x^2+y^2=4$

The equations $y=x^2$ & $x^2+y^2=4$ are implicit equations of those curves.

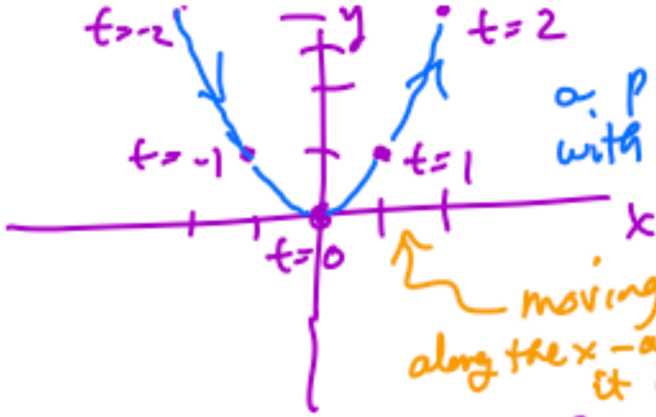
More convenient for higher dimensions: parametrized curves. Use a variable t for time — we will give a formula for position at time t.

Example $y=x^2$

Let $x=t \Rightarrow y=t^2$.

Curve equation $\alpha(t) = (\underset{x}{t}, \underset{y}{t^2})$

We could parametrize any $y=f(x)$ as $\alpha(t)=(t,f(t))$.



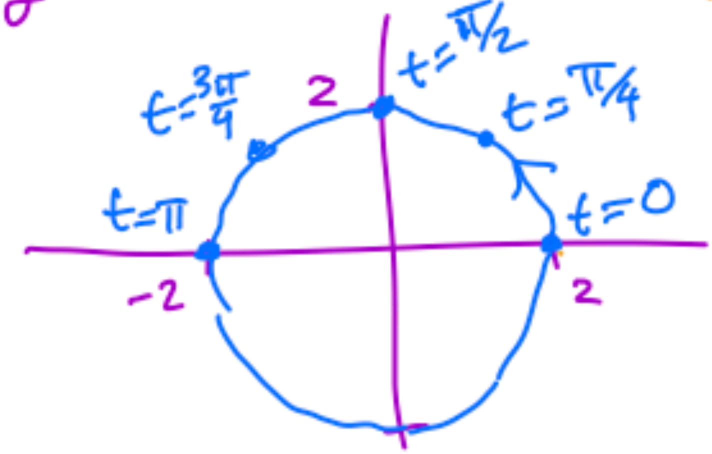
a parabola but with a particle moving along it.

moving at a constant speed along the x-axis, but along the curve, it is faster as t gets large.

Another example $x^2+y^2=4$

$$x = 2\cos t$$
$$y = 2\sin t$$

the particle is moving at constant speed here.



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alternative for $x^2+y^2=4$

$$y = \pm\sqrt{4-x^2}$$

$$\beta(t) = \left(t, \sqrt{4-t^2}\right)$$

$0 \le t \le 2$

$\frac{1}{4}$ circle.

There are many different parametrizations of the same curve.